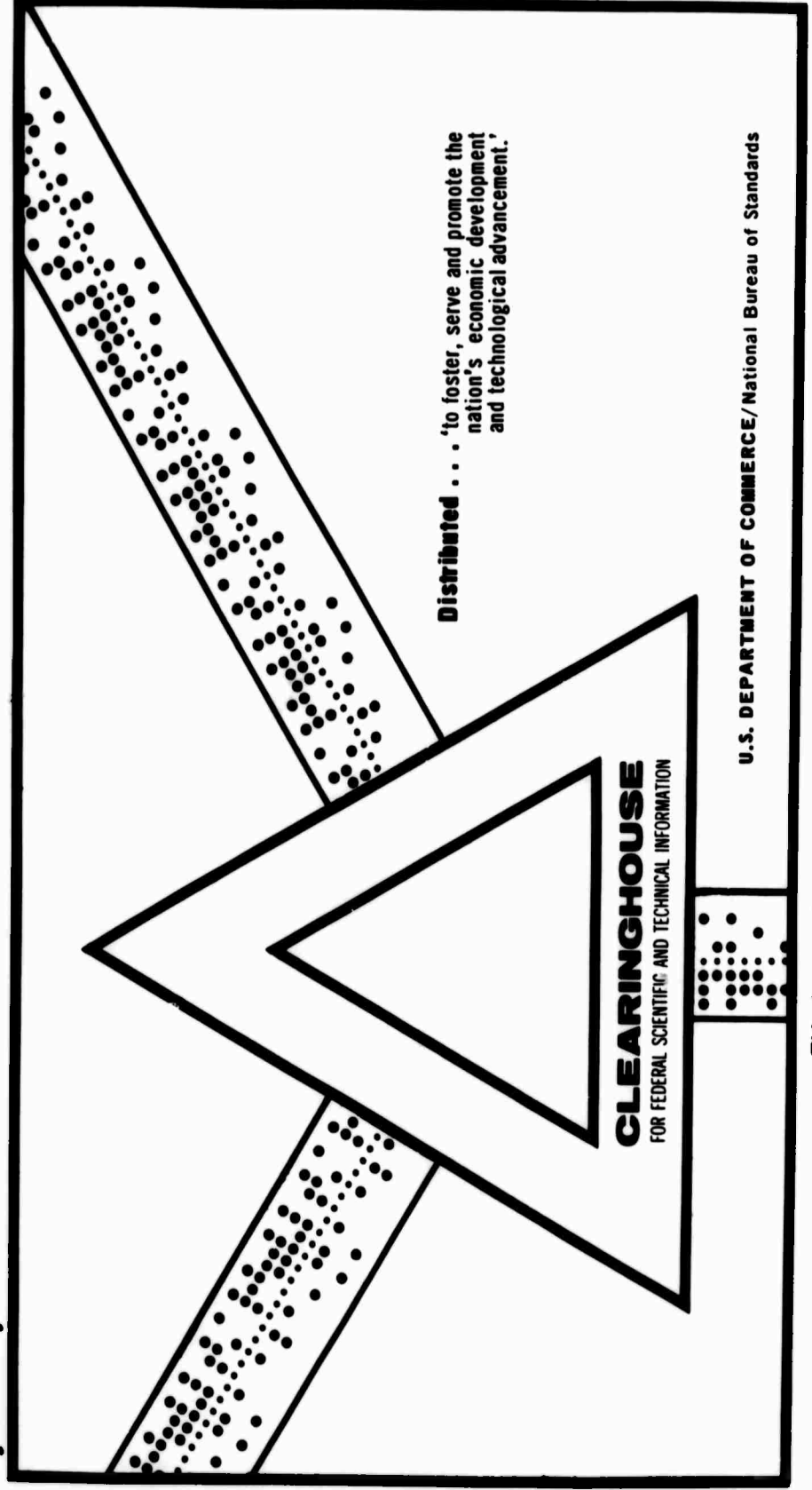AD 700 046

APPROXIMATE SOLUTION OF BOUNDARY LAYER HEAT TRANSFER EIGEN-VALUE PROBLEMS WITH APPLICATIONS

Nelson H. Kemp

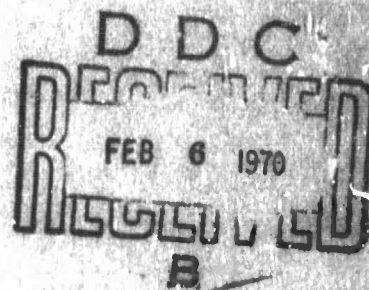Avco-Everett Research Laboratory
Everett, Massachusetts

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RESEARCH REPORT 337
JANUARY 1970

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APPROXIMATE SOLUTION OF BOUNDARY LAYER HEAT TRANSFER
EIGENVALUE PROBLEMS WITH APPLICATIONS*†

by

Nelson H. Kemp††

January 1970

AVCO EVERETT RESEARCH LABORATORY
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Everett, Massachusetts

FOREWORD

Publication of this report does not constitute Air Force approval of the report's findings or conclusions; it is published only for the exchange and stimulation of ideas.

Lt. R. W. Padfield, USAF
Project Officer,
Environmental Technology Branch,
SMYSE

ABSTRACT

We study three eigenvalue problems of boundary layer theory first introduced by Fox and Libby,[1] and pursued further by Kotorynski.[2] By completing Kotorynski's approximate analysis, we exhibit approximate eigenvalues and norms. With these, the convergence of the eigenfunction series solutions of two heat transfer problems can be investigated: (a) Flat plate with one constant wall temperature region followed by another. (b) Flat plate with a constant wall temperature region followed by an insulated wall. The series for problem (b) is found to converge everywhere, including at the station of change in the boundary condition. The series for problem (a) does not converge at that station, and convergence for both appears slow near that station. Then we improve Kotorynski's approximation and use the WKB method to obtain simple formulas for the eigenvalues, which prove very accurate, and enable the eigenfunctions to be found by a simple forward integration. Finally, we study problem (b) in more detail, and obtain an exact solution by a numerical finite difference procedure. This shows the eigenfunction series solution to be very slowly convergent near the beginning of the insulated region. The solution worked out by Durgin[9,10] based on Lighthill's[4] approximate method, is very accurate there. In addition, it is within 10% of the numerical solution everywhere. It is undoubtedly the most convenient representation of the solution, agreeing both with the new numerical solution, and with Durgin's experiments.

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I. INTRODUCTION

Fox and Libby[1] have developed perturbation solutions to the momentum and energy equations of laminar boundary layer theory. The momentum solutions are perturbations of the Blasius flat plate solution. Among the energy solutions are those for initial value problems with the Blasius velocity field and either constant wall enthalpy, or zero heat transfer, downstream of the initial station.

In their investigations, Fox and Libby formulated three eigenvalue problems: (1) For the perturbation of the momentum equation caused by initial values differing from the Blasius profile; (2) for the initial value energy problem with constant wall enthalpy; (3) for the initial value energy problem with zero heat transfer. In each case they found the first ten eigenvalues, norms, and eigenfunctions by a numerical searching procedure. Subsequently Libby[2] recalculated the eigenvalues and norms for problem (1) and added the next ten sets of values.

Kotorynski[3] has considered these three eigenvalue problems analytically as irregular Sturm-Liouville systems. He reduced them to normal form and approximated the coefficient function by the sum of its principal terms near the wall, and far from the wall. He pointed out that the resulting equations could be solved in terms of Laguerre polynomials, and found approximate expressions for the eigenvalues. The resulting eigenvalues are not particularly accurate, the twentieth for problem (1) being 4.3% higher than Libby's computed value. Kotorynski did not use the boundary conditions to

find explicitly the approximate eigenfunctions or the norms, to compare with Fox and Libby's numerical values.

The purpose of the present paper is three-fold: (1) To extend the approximate eigenvalue results of Kotorynski to include the approximate eigenfunctions and norms, and investigate the convergence of some solutions obtained by Fox and Libby, using these eigenvalues and norms. (2) To apply the results of a WKB approximation to obtain another approximate formula for the eigenvalues which is extremely accurate indeed, and should enable numerical calculation of the eigenfunctions and norms with little difficulty in a minimum of computer time. (3) To discuss the insulated wall problem of Fox and Libby further, comparing the eigenfunction series solution with several other approximate solutions, and with a new exact finite difference solution. This comparison emphasizes the slow convergence of the eigenfunction solution near the initial station, and the remarkable accuracy of a solution obtained using Lighthill's approximate method. [4]

II. FORMULATION

In similarity variables ξ, η the momentum and energy equations for unit Prandtl number and constant density-viscosity product are

$$f_{\eta\eta\eta} + f f_{\eta\eta} - 2\xi (f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}) = 0 \tag{2.1a}$$

$$g_{\eta\eta} + f g_\eta - 2\xi (f_\eta g_\xi - f_\xi g_\eta) = 0 \tag{2.1b}$$

We are interested in initial value problems, where the non-dimensional velocity and enthalpy profiles f_η, g are given at some initial streamwise station ξ_i by $F^*(\eta)$ and $G^*(\eta)$. The wall boundary conditions are the usual ones of zero velocity and either constant temperature (problem (2)) or zero heat transfer (problem (3)). Thus

$$
\begin{aligned}
\eta = 0: \quad & f = f_\eta = 0; \qquad && g = g_w \quad \text{for problem 2} \\
& && g_\eta = 0 \quad \text{for problem 3} \\
\eta \to \infty: \quad & f_\eta \to 1, \qquad && g \to 1 \\
\xi = \xi_i: \quad & f_\eta = F^*(\eta), \qquad && g = G^*(\eta)
\end{aligned}
\right\} \tag{2.2}
$$

For the momentum problem (1), we perturb about the Blasius solution $f_0(\eta)$:

$$f_0''' + f_0 f_0'' = 0, \qquad f_0(0) = f_0'(0) = 0, \qquad f'(\infty) = 1 \tag{2.3}$$

With the assumed solution as

$$f = f_0(\eta) + \sum_1^\infty f_k(\xi, \eta); \qquad f_{k+1} \ll f_k \tag{2.4}$$

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the perturbation equations of (2. 1a) become for each f_k, $k = 1, 2 \ldots$

$$\left[f_{\eta\eta\eta} + f_o f_{\eta\eta} + f_o'' f - 2\xi(f_o' f_{\eta\xi} - f_o'' f_\xi) \right]_k = F_k \qquad (2.5)$$

where F_k is dependent on the previous perturbation terms f_j, $o < j < k$. In particular, for f_1, $F_1 = 0$ and (2. 5) is homogenous. In addition, for higher perturbations, the homogenous solution can be used to find the particular integral. So we will focus on the solution of the homogenous equation, which will lead us to an eigenvalue problem when we separate variables.

We therefore seek solutions of the homogenous equation associated with (2. 5) in the form (dropping the subscript k).

$$f = \sum_o^\infty A_n \, (\xi/\xi_i)^{-\lambda_n^{(1)}/2} \, N_n(\eta) \qquad (2.6)$$

and find N_n and $\lambda_n^{(1)}$ must satisfy

$$N''' + f_o N'' + \lambda f_o' N' + (1 - \lambda) f_o'' N = 0 \qquad (2.7)$$

Since f_o already satisfies all the boundary conditions according to (2. 2) and (2. 3) the boundary conditions on N are homogeneous:

$$N(0) = N'(0) = 0, \qquad N'(\infty) = 0 \qquad (2.8)$$

As discussed in Part I of Fox and Libby,[1] (2. 7) and (2. 8) define an eigenvalue problem, the eigenvalues to be chosen so that N' approaches zero exponentially as $\eta \to \infty$. (If only algebraic decay were required, there would be solutions for any λ.) To make the functions N well-defined, Fox and Libby imposed the scaling condition

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$$N''(0) = 1 \tag{2.9}$$

For problems (2) and (3), f is taken to be unperturbed Blasius function $f_o(\eta)$ in (2.1b) and the solution for g is assumed as

$$g = g_o(\eta) + g^*(\xi, \eta) \tag{2.10}$$

leading to the homogeneous equation

$$g^*_{\eta\eta} + f_o g^*_\eta - 2\xi f_o' g^*_\xi = 0 \tag{2.11}$$

The function $g_o(\eta)$ must satisfy

$$g_o'' + f_o g_o' = 0 \tag{2.12a}$$

and we take it to satisfy also the boundary conditions (2.2) on $\eta = 0$, ∞:

$$\eta = 0: \quad \begin{array}{ll} g_o = g_w & \text{for problem (1)} \\ g_o' = 0 & \text{for problem (2)} \end{array} \left.\vphantom{\begin{array}{l} a \\ a \\ a \end{array}}\right\} \tag{2.12b}$$
$$\eta \to \infty \quad g_o \to 1$$

It is thus completely determined, and will make a contribution to the initial profile to be satisfied by g^*. However, as a function of η, g^* satisfies homogenous boundary conditions, since g_o satisfies all the inhomogenous ones, according to (2.2) and (2.12b).

The homogenous equation (2.11) has separable solutions of the form

$$g^* = \sum_0^\infty B_n (\xi/\xi_o)^{-\lambda_n^{(2,3)}/2} H_n^{(2,3)}(\eta) \tag{2.13}$$

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where $\lambda_n^{(2,3)}$ and $H_n^{(2,3)}$ satisfy

$$H'' + f_o H' + \lambda f_o' H = 0 \qquad (2.14)$$

The boundary conditions on H for the constant enthalpy wall, problem (2), are (including a convenient scaling)

Problem (2): $\quad H^{(2)}(0) = 0, \ H^{(2)}(\infty) = 0; \ [H^{(2)'}(0) = 1] \qquad$ (2.15a)

while for the adiabatic wall, problem (3), they are

Problem (3): $\quad H^{(3)'}(0) = 0, \ H^{(3)}(\infty) = 0; \ [H^{(3)}(0) = 1] \qquad$ (2.15b)

These are two eigenvalue problems with the same differential equation but different boundary conditions at the wall. Again the eigenvalues are determined by the condition that $H \to 0$ exponentially as $\eta \to \infty$. These problems were discussed by Fox and Libby in Part 2.[1]

Problem (1) has also been studied by Stewartson,[5] who showed that $\lambda_o^{(1)} = 2$ is an eigenvalue with eigenfunction $N_o = (\eta f_o' - f_o)/f_o''(0)$. For problem (3), Fox and Libby point out that $\lambda_o^{(3)} = 1$, $H_o^{(3)} = f_o''/f_o''(0)$, satisfies (2.14) and (2.15b), yielding one explicit solution. No corresponding result for either λ or H is known for problem (2).

The standard Sturm-Liouville form of (2.7) is obtained by recognizing that f_o' is a solution (though it does not satisfy the boundary conditions) so the introduction of a new variable $u = (N/f_o')'$ yields

$$[(f_o'^3/f_o'') u']' + [\lambda f_o'^4/f_o'' - f_o f_o'^2] u = 0 \qquad (2.16)$$

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which shows the weight function explicitly. The orthogonality condition
is then

$$\int_{0}^{\infty} (f_o'^4/f_o'') \, u_n \, u_m \, d\eta \; = C_n^{(1)} \delta_{mn} \qquad (2.17)$$

with $C_n^{(1)}$ the norms for problem (1).

For problems (2) and (3), (2.14) can be put into standard form by
dividing by f_o'' and using (2.3), to yield

$$(H'/f_o'')' + (\lambda \, f_o'/f_o'') \, H = 0 \qquad (2.18)$$

and the orthogonality condition and norms are

$$\int_{0}^{\infty} (f_o'/f_o'') \, H_n H_m d\eta = C_n^{(2,3)} \, \delta_{mn} \qquad (2.19)$$

The norms corresponding to the exact eigenfunctions quoted above are easily
found by integration to be

$$C_o^{(1)} = C_o^{(3)} = 1/(2a^2) = 2.267, \quad a \equiv f_o''(0) = 0.4696 \quad (2.20)$$

which both agree with Fox and Libby's calculated values.

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III. EXTENSION OF KOTORYNSKI'S APPROXIMATE
ANALYSIS

Kotorynski[3] studied these Sturm-Liouville problems by reducing them to the form in which the coefficient of the eigenvalue is unity. For problem (1), this transformation is

$$w = (f_o'^7/f_o''^2)^{1/4} \, u, \qquad t = \int_o^\eta (f_o')^{1/2} \, d\eta \qquad (3.1)$$

which reduces (2.16) to

$$d^2w/dt^2 + [\mu - q(t)] \, w = 0, \qquad (3.2a)$$

$$\mu = \lambda - \frac{1}{2}, \qquad q = \frac{3}{4} \, \frac{f_o f_o''}{f_o'^2} + \frac{7}{16} \, \frac{f_o''^2}{f_o'^3} + \frac{1}{4} \, \frac{f_o^2}{f_o'} \qquad (3.2b)$$

For problems 2 and 3, the corresponding transformation is

$$v = (f_o'/f_o''^2)^{1/4} \, H, \qquad t = \int_o^\eta (f_o')^{1/2} \, d\eta, \qquad (3.3)$$

reducing (2.18) to

$$d^2v/dt^2 + \left[\mu - p(t)\right] \, v = 0 \qquad (3.4a)$$

$$\mu = \lambda - \frac{1}{2}, \qquad p = -\frac{1}{4} \, \frac{f_o f_o''}{f_o'^2} - \frac{5}{16} \, \frac{f_o''^2}{f_o'^3} + \frac{1}{4} \, \frac{f_o^2}{f_o'} \qquad (3.4b)$$

The similarity of the equations is strongly evident, the functions p, q differing only in the coefficients of the first two terms.

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The behavior of the transformations and the functions p and q is determined by the well-known properties of the Blasius function f_o, which are

$$\eta \to 0: \quad f_o \to a\eta^2/2, \quad f_o' \to a\eta, \quad f_o'' \to a, \quad a = 0.4696 \qquad (3.5a)$$

$$\eta \to \infty: \quad f_o \to \eta - \beta, \quad f_o' \to 1, \quad f_o'' \to Ae^{-(\eta-\beta)^2/2}, \qquad (3.5b)$$

$$\beta \equiv \int_0^\infty (1-f_o') \, d\eta = 1.217$$

This requires that

$$\eta \to 0: \quad t \to 2(a\eta^3)^{1/2}/3 \qquad (3.5c)$$

$$\eta \to \infty: \quad t \to \eta - \int_0^\eta \left[1 - (f_o')^{1/2}\right] d\eta \to \eta - \int_0^\infty \left[1 - (f_o')^{1/2}\right] d\eta \qquad (3.5d)$$

Thus we find from (3.2) and (3.4) that

$$t \to 0: \quad q \to 7/(36t^2), \quad p \to -5/(36t^2) \qquad (3.6a)$$

$$t \to \infty: \quad q \to p \to (t-\delta)^2/4, \quad \delta \equiv \int_0^\infty \left[(f_o')^{1/2} - f_o'\right] d\eta \qquad (3.6b)$$

Kotorynski proposed a uniform approximation to p and q be obtained by using the terms dominant at $t \to 0, \infty$, since the composite function would have the correct behavior at both ends. Thus (3.2a) and (3.4a) are approximated by

$$\frac{d^2w}{dt^2} + \left[\mu - \frac{7}{36t^2} - \frac{(t-\delta)^2}{4}\right] w = 0,$$

$$\frac{d^2v}{dt^2} + \left[\mu + \frac{5}{36t^2} - \frac{(t-\delta)^2}{4}\right] v = 0 \qquad (3.7a, b)$$

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Actually, Kotorynski did not include the δ term in these equations. Without it, they are solvable, as he showed. With it, they are apparently not solvable in terms of known functions. However, the application of the WKB method for finding the eigenvalues is possible even when δ is retained, and the δ terms greatly improve the accuracy of the resulting eigenvalues, compared to those found by Kotorynski, as we shall see in Section V.

A comparison between the exact functions q and p of (3. 2b) and (3. 4b) and the approximations given in (3. 7a, b) is shown in Fig. 1. They are plotted against η, which is related to t by (3. 1). The approximations are shown for both δ = 0 and δ = 0. 425, which is the value given in Stewartson. [5] For q, there is a small difference between the exact function and the approximation with δ ≠ 0 for $0. 6 \leq \eta \leq 3. 2$, but they agree quite well elsewhere. If δ = 0, the approximation is substantially in error for $\eta > 3$. For p, there is very little error in the approximation when δ ≠ 0 to the scale of Fig. 1; if δ = 0 there are again substantial errors for $\eta > 3$. It is clear that using the correct asymptotic representation of f_o makes a large improvement in the accuracy of approximation of p and q.

Before the approximate equations can be solved, the boundary conditions must be stated in terms of w and v. If one uses the boundary conditions (2. 8) and (2. 9), traces through the transformation to u, (2. 16), and to w, t (3. 1), and uses (3. 5c), one finds

$$t \to 0: \quad w \to (3t/2)^{7/6}/2 \, a^{1/3} \tag{3. 8a}$$

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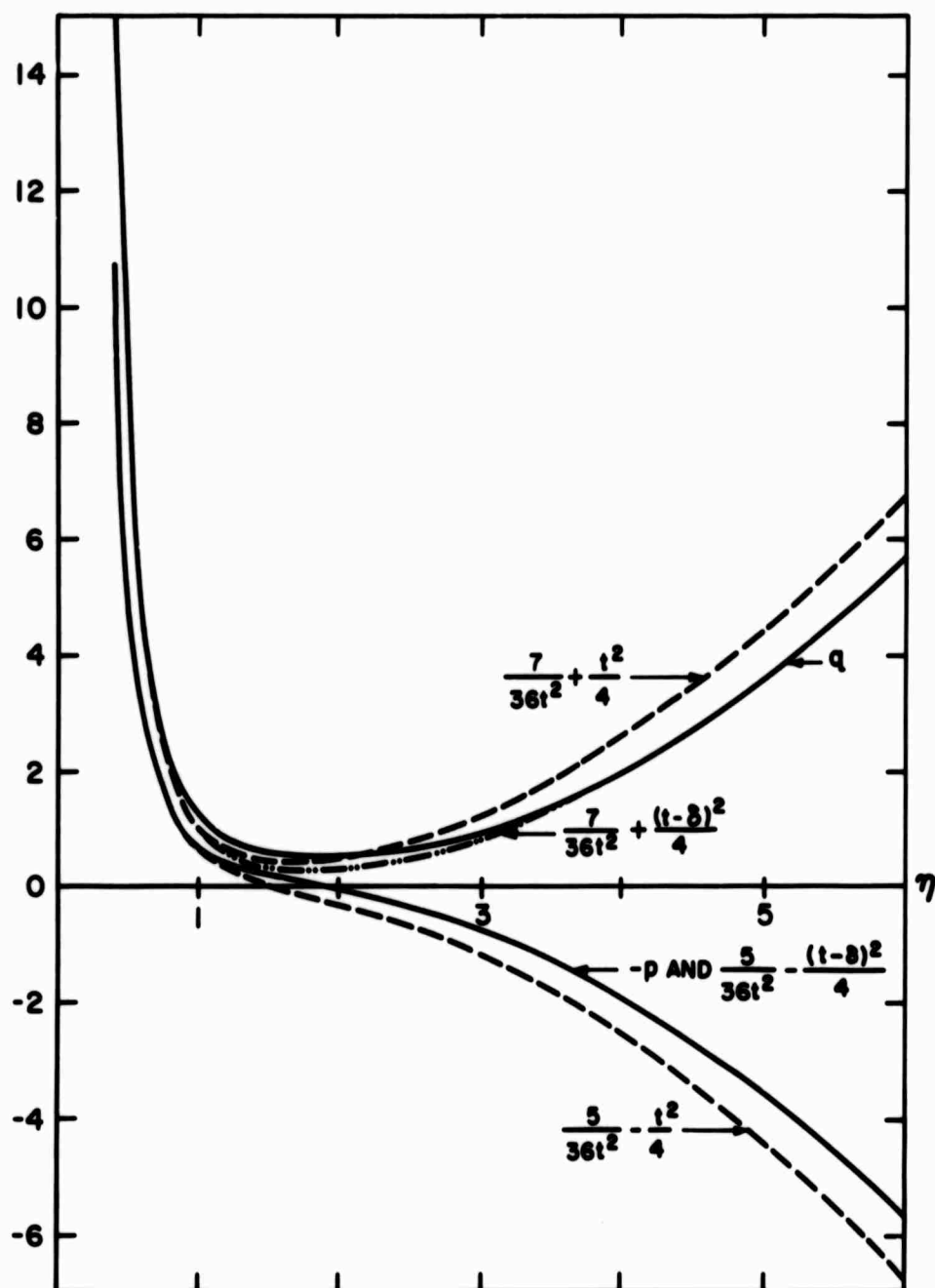
Fig. 1 Comparison of exact and approximate representations of the functions q and p in the normal form of the Sturm-Liouville equations.

Fox and Libby[1] have shown that the asymptotic form of N' is

$$N' \sim (\eta - \beta)^{\lambda - 1} \exp\left[-(\eta - \beta)^2/2\right]$$

When this is transformed to w, t using (3. 5b, d), we find

$$t \to \infty: \quad w \sim (t-\delta)^{\lambda - 1} \exp\left[-(t-\delta)^2/4\right] \tag{3. 8b}$$

Similarly, the boundary conditions (2. 15a) and (2. 15b) for $v^{(2)}$ and $v^{(3)}$ yield, respectively, for problems 2 and 3:

$$t \to 0: \quad t^{-1/6} v^{(2)} \to 0, \quad (3 a t/2)^{1/6} \frac{dv^{(2)}}{dt} - \frac{a v^{(2)}/4}{(3 a t/2)^{5/6}} \to \frac{1}{a^{1/2}} \tag{3. 9a}$$

$$v^{(3)} \to (3 t/2 a^2)^{1/6}, \quad (3 at/2)^{1/6} \frac{dv^{(3)}}{dt} \to \frac{a^{-1/6}/4}{(3 t/2)^{2/3}} \tag{3. 9b}$$

$$t \to \infty: \quad v \sim (t-\delta)^{\lambda - 1} \exp\left[-(t-\delta)^2/4\right] \tag{3. 9c}$$

We may now proceed to the solution of (3. 7) for $\delta = 0$. A change of variable $t = 2^{1/2} x$ permits them both to be written

$$d^2 y/dx^2 + [2\mu - x^2 + (4^{-1} - a^2)/x^2] = 0$$

Problem 1: $a = \pm 2/3$; Problem 2, 3: $a = \pm 1/3$ \hfill (3. 10)

and a solution of this equation is given in Ref. 6, p. 781, Eq. (22. 6. 18) as

$$y \sim e^{-x^2/2} x^{a+1/2} L_n^{(a)}(x^2),$$

$$2\mu = 2\lambda - 1 = 4n + 2a + 2, \quad n = 0, 1, 2, \ldots \tag{3. 11}$$

where $L_n^{(a)}(x^2)$ is the generalized Laguerre polynomial defined in Ref. 6, p. 775, Eq. (22. 3. 9), as an n-th order polynomial in x^2. In

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terms of t, we may take as our solutions*

$$\left\{ w_n, \ v_n^{(2)}, \ v_n^{(3)} \right\} = \left\{ D_n^{(1)}, \ D_n^{(2)}, \ D_n^{(3)} \right\} e^{-t^2/4} \ t^{a+1/2} L_n^{(a)}(t^2/2) \quad (3.12)$$

with the D_n to be determined by the boundary conditions.

First we notice that this solution satisfies the boundary condition as $t \to \infty$; in fact (3.12) becomes proportional to $e^{-t^2/4} \ t^{2n + a + 1/2}$ which, from (3.11), agrees exactly with (3.8b) and (3.9c). Second, we find the behavior of w, v and their derivatives as $t \to 0$ from Ref. 6, p. 775, Eq. (22.3.9):

$$t \to 0: \ w, \ v \sim t^{a+1/2} \ [\Gamma(n + a + 1)/n! \ \Gamma(a + 1) + O(t^2)] \quad (3.13a)$$

$$w', \ v' \sim (a + 1/2) \ t^{a-1/2} \ [\Gamma(n + a + 1)/n! \ \Gamma(a + 1) + O(t^2)] \quad (3.13b)$$

Third, the eigenvalues are given in terms of a by (3.11). For each problem, there are two possible values of a, Eq. (3.10), only one of which will satisfy the boundary conditions at $t = 0$.

Kotorynski did not actually exhibit the resulting approximate eigenfunctions which satisfy the boundary conditions by evaluating the constants D_n, nor did he find the norms C_n. However, the expression for the norms is easily found from the orthogonality properties of the $L_n^{(a)}$. Comparison of the definitions (2.17) and (2.19) with the transformations (3.1) and 3.3) show that we need the integral

$$I \equiv \int_0^\infty e^{-t^2/2} \ t^{2a+1} \ \left[L_n^{(a)} (t^2/2) \right]^2 \ dt$$

---

*The solution given in (3.12) can be compared with Kotorynski's Eq. (3.5). Our a is his $2a$, but his argument of L does not have the factor 1/2. This represents either a different definition of L (which he does not give) or an error in his formula.

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But Eq. (22.2.12) on p. 775 of Ref. 6 shows that with $z = t^2/2$,

$$2^{-a} I = \int_0^\infty e^{-z} z^a \left[L_n^{(a)}(z) \right]^2 dz = \Gamma(n + a + 1)/n!$$

Thus the norms are related to the coefficients D_n by

$$C_n = 2^a D_n^2 \Gamma(n + a + 1)/n! \tag{3.14}$$

The approximate solutions can now be found by choosing D_n to satisfy the boundary conditions (3.8a), (3.9a, b) as $t \to 0$, using the asymptotic forms from (3.13a, b). In the process of satisfying the boundary conditions, one finds which one of the two a values for each problem, Eq. (3.10), is the correct one; this determines the approximate value of the eigenvalues from Eq. (3.11). The norms follow from D_n by (3.14).

The results of applying this process can be expressed compactly:

$$\left\{ a^{(1)}, a^{(2)}, a^{(3)} \right\} = \left\{ 2/3, 1/3, -1/3 \right\} \tag{3.15a}$$

$$\mu_n^{(i)} + 1/2 = \lambda_n^{(i)} = 2n + a^{(i)} + 3/2 \tag{3.15b}$$

$$\left\{ D_n^{(1)}, D_n^{(2)}, D_n^{(3)} \right\} = \left(\frac{3}{2a^2} \right)^{a^{(i)}+1/2} \frac{n! \Gamma(a^{(i)} + 1)}{\Gamma(a^{(i)} + n + 1)} \left\{ \frac{a^2}{2}, a, 1 \right\} \tag{3.16}$$

$$C_n^{(i)} = \frac{n! \Gamma(a^{(i)} + 1)}{\Gamma(a^{(i)} + n + 1)} \qquad C_o^{(i)} = \frac{n}{n + a^{(i)}} \quad C_{n-1}^{(i)} \tag{3.17a}$$

$$\left\{ C_o^{(1)}, C_o^{(2)}, C_o^{(3)} \right\} = 2^{a^{(i)}} \left(\frac{3}{2a^2} \right)^{2a^{(i)}+1} \Gamma(a^{(i)} + 1) \left\{ \frac{a^4}{4}, a^2, 1 \right\}$$

$$= \left\{ 1.52, 6.05, 2.04 \right\} \tag{3.17b}$$

This completes the solution of the three problems when q, p are approximated by the sum of their asymptotic expressions as t → 0 and ∞ , and the constant δ in the latter asymptote is taken to be zero.

IV. CONVERGENCE OF HEAT TRANSFER SOLUTIONS

We will now consider briefly the solutions of problems 2 and 3, concentrating on the convergence of the series solutions. For this purpose, the approximate eigenvalues and normals obtained in the previous section will be used. Although they are not particularly accurate for moderate n, they become accurate for large n, and so are useful for convergence studies.

In problem 2, Fox and Libby[1] consider a flat plate which is held at a fixed enthalpy g_w, up to ξ_i and then changed to g_{wo}. A function g_o which satisfies (2.12) is $g_o = g_{wo} + (1-g_{wo}) f_o'$. Then g* is given by (2.13) and (2.14) with boundary conditions (2.15a). The coefficients $B_n^{(2)}$ are determined by the initial condition on g, which is that up to ξ_i, g is the Crocco integral $g = g_{wl} + (1-g_{wl}) f_o'$. The initial condition on g* then determines $B_n^{(2)}$ as an expansion coefficient:

$$(g_{wl} - g_{wo})(1-f_o') = \sum_0^\infty B_n^{(2)} H_n^{(2)}$$

Using the orthogonality condition (2.19) as well as the Sturm-Liouville form of the equation for H, (2.18), the integral can be evaluated to give

$$B_n^{(2)} = (g_{wl}-g_{wo})/\left(\lambda_n^{(2)} \, a \, C_n^{(2)}\right) \qquad a \equiv f_o''(0) = 0.4696$$

The heat transfer at the wall is found using this in (2.13) and differentiating (2.10):

$$g_{\eta w} = (1-g_{wo}) a + (g_{wl}-g_{wo}) a^{-1} \sum_0^\infty \left(\lambda_n^{(2)} \, C_n^{(2)}\right)^{-1} (\xi/\xi_i)^{-\lambda_n^{(2)}/2} \qquad (4.1)$$

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This agrees with Fox and Libby, Part 2, Eqs. (2. 26b) and (2. 27), who take an initially adiabatic wall so $g_{wl} = 1$.

In problem 3, the wall is still held at g_{wl} up to ξ_i, but then changes to an adiabatic wall, $g_\eta = 0$. In this case $g_o = 1$, and the initial condition on g is the same Crocco integral as above. Therefore, the initial condition for this problem, using again the solution (2. 13) for g^*, is

$$-(1-g_{wl}) \ (1-f_o') = \sum_o^\infty B_n^{(3)} \ H_n^{(3)}$$

The coefficient is found the same way as before, although the result differs, since the H satisfies different boundary conditions.

$$B_n^{(3)} = -(1-g_{wl}) / (\lambda_n^{(3)} \ C_n^{(3)})$$

Using this in (2. 13), the wall temperature is found from (2. 10) as

$$(1-g_w)/(1-g_{wl}) = \sum_o^\infty (\lambda_n^{(3)} \ C_n^{(3)})^{-1} \ (\xi/\xi_i)^{-\lambda_n^{(3)}/2} \tag{4. 2}$$

This agrees with Fox and Libby, Part 2, Eqs. (2. 30b) and (2. 31), when a misprinted subscript o is changed to 1 in the latter equation.

We now consider the convergence of the sums in (4. 1) and (4. 2), using (3. 15) and (3. 17) for λ_n and C_n. The ratio of the n+1 to the n term, for large n, becomes

$$r = \left[1 + (a^{(i)}-1)/n\right] \xi_i/\xi$$

For $\xi > \xi_i$, this converges by the ratio test, although the convergence is very slow near ξ_i. For $\xi = \xi_i$, at the initial plane, the series only converges if $a^{(i)} - 1 < -1$, i. e., $a^{(i)} < 0$. Thus the series (4. 2) for

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problem 3 converges at $\xi = \xi_i$, since $a^{(3)} = 1/3$, but the series (4.1) for problem 2 diverges, since $a^{(2)} = 1/3$. This is in accord with the indications of the numerical results in Fox and Libby, Part 2, where Fig. 4 shows $g_{\eta w}$ rising steeply near the initial plane, while Fig. 5 has a finite value for g_w there.

Notice that if one differentiates (4.2) with respect to ξ, one brings down a λ_n, cancelling that term in the denominator. Then the ratio of terms for large n is

$$r = (1 + a^{(i)}/n)\, \xi/\xi_i$$

which still yields convergence for $\xi > \xi_i$, but now shows that problem 3 has a divergent streamwise derivative of g_w, since $a^{(3)} = -1/3 > -1$. This would indicate that the curves of Fig. 5 in Fox and Libby, Part 2, should have an infinite slope at the initial plane a property that their solution does not exhibit in the Figure. This is likely caused by the use of only 10 terms of a slowly - converging series. Section VI has a fuller discussion of problem 3, where it is shown that the next 10 terms make a substantial contribution, and that the slope at the initial plane is indeed very steep and probably infinite.

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V.   ESTIMATION OF EIGENVALUES BY THE WKB METHOD

The eigenvalues of (3. 7a, b) can be estimated by a technique well
known in quantum mechanics for handling the Schrödinger equation.   The
advantage of this over Kotorynski's approach is that the correct behavior
for large t, which retains the constant δ, can be used.   This leads to a
much more accurate estimate, but does not easily give approximate
eigenfunctions.   However, once accurate eigenvalues are known, the
numerical integration of the exact equations (2. 7) or (2. 14) presents
little difficulty.   The difficulties encountered by Fox and Libby are
primarily associated with the necessity of finding the eigenvalues by
trial and error.

The idea of the WKB method is to solve (3. 7a, b) in three regions.
One is where t is very large, so the t^{-2} term is negligible and $(t-\delta)^2/4 > \mu$;
in this region there are exponential-type solutions.   A second region is
where t is still large, so t^{-2} is neglected, but $(t-\delta)^2/4 < \mu$ so the solutions
have sinusoidal character.   These two solutions are matched at the turning
point t_o where

$$(t_o - \delta)^2/f = \mu \qquad\qquad (5.1)$$

so as to eliminate the increasing exponential term, and make the solution
continuous.   (This matching procedure is described in detail in Ref. 6,
for example.)   To satisfy the boundary conditions at t = 0, a third solution
is found in the region where the t^{-2} term dominates $(t-\delta)^2/4$.   Once the
boundary conditions are satisfied with this solution, its asymptotic

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expansion is joined to the solution in the second region; this joining gives a condition for determining the eigenvalues.

This procedure can be carried out in general for (3. 7a, b) and the appropriate boundary conditions (3. 8a) or (3. 9a, b). However the result is already available in a quadrature formula in Ref. 7, as pointed out by Kotorynski, who used it with $\delta = 0$ to verify his approximate eigenvalues (3. 15). Titchmarsh[7] gives, on page 151, Eq. (7. 76), a formula which for (3. 7a, b) becomes

$$\frac{1}{\pi} \int_0^{t_o} \left[\mu_n - \frac{(t-\delta)^2}{4} \right]^{1/2} dt = \frac{a+1}{2} + n + O\left(\frac{1}{n}\right), \quad n = 0, 1, 2, \ldots$$

where t_o is defined in (5. 1). By integrating the left side, we obtain an equation for μ_n.

Carrying at the integration by a trignometric substitution we find

$$\frac{\mu_n}{2} \left\{ 1 + \frac{2}{\pi} \sin^{-1}\left(\frac{\delta}{2\mu_n^{1/2}}\right) + \frac{2}{\pi}\left(\frac{\delta}{2\mu_n^{1/2}}\right)\left[1 - \left(\frac{\delta}{2\mu_n^{1/2}}\right)^2\right]^{1/2} \right\} = \frac{a+1}{2} + n + O\left(\frac{1}{n}\right)$$

This shows $\mu_n = O(n)$, so we may keep terms in $\left\{ \ \right\}$ through $O(\mu_n^{-3/2})$. The result is

$$\mu_n + \frac{2\delta}{\pi} \mu_n^{1/2} - \frac{\delta^3}{12\pi} \mu_n^{-1/2} = a + 1 + 2n + O\left(\frac{1}{n}\right)$$

This is a cubic in $\mu_n^{1/2}$, but the last term on the left is quite small, because Stewartson has given[5]

$$\delta \equiv \int_0^\infty \left[\left(f_o'\right)^{1/2} - f_o'\right] d\eta = 0. 425$$

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Therefore, we will ignore this term, and solve the quadratic to get

$$\mu_n^{1/2} = (\lambda_n - 1/2)^{1/2} = -\delta/\pi + \left[(\delta/\pi)^2 + a + 1 + 2n\right]^{1/2}, \quad n = 0, 1, 2, \ldots \quad (5.3)$$

For $\delta = 0$, (5.3) agrees exactly with (3.15), which was the result obtained by Kotorynski. The improvement obtained by including the δ term is remarkable, as shown in Table I, where the exact results of Fox and Libby are compared with those of Kotorynski, Eq. (3.15), the present results, Eq. (5.3), and some additional exact results for problem 3, obtained by the present author. Even at the lowest eigenvalues, the error is only 7.4% for problem 1, 1.1% for problem 2, and 2.1% for problem 3. This accuracy is a pleasant surprise since the WKB method is designed for the higher eigenvalues. The errors are less than 1% beginning with the fourth eigenvalue for problem 1, and the second for problems 2 and 3; they are less than 0.1% after the thirteenth eigenvalue for problem 1 and the fourth for problems 2 and 3. It is clear that the use of (5.3) will enable the numerical solution of (2.7) or (2.14) to be performed in one forward integration from $\eta = 0$.

If Eqs. (3.7a, b) could be solved exactly with the δ terms included, presumably very accurate norms could also be found, but the present author has been unable to find such a solution in terms of known functions. The norms obtained by the Kotorynski approximation, (3.17), are shown for problem 3 in Table II, together with exact values calculated by the present author. The errors vary from 10% at $n = 0$ to 1% at $n = 19$. (The first 10 exact values of C_n in Table II differ in the second decimal place from those given on p. 438 of Fox and Libby,[1] Part 2. They are believed to be

TABLE I.1

Eigenvalues for Problem 1, $\alpha = 2/3$

n	$\lambda_n(\text{exact})^2$	$\lambda_n(\text{Eq. 3.15})^3$	$\lambda_n(\text{Eq. 5.3})$	n	$\lambda_n(\text{exact})^2$	$\lambda_n(\text{Eq. 3.15})^3$	$\lambda_n(\text{Eq. 5.3})$
0	2	2.167	1.852	10	20.984	22.167	20.943
1	3.774	4.167	3.684	11	22.924	24.167	22.886
2	5.627	6.167	5.558	12	24.862	26.167	24.832
3	7.517	8.167	7.453	13	26.865	28.167	26.780
4	9.408	10.167	9.361	14	28.755	30.167	28.729
5	11.328	12.167	11.278	15	30.711	32.167	30.680
6	13.252	14.167	13.202	16	32.667	34.167	32.633
7	15.171	16.167	15.132	17	34.620	36.167	34.587
8	17.099	18.167	17.066	18	36.567	38.167	36.542
9	19.040	20.167	19.003	19	38.520	40.167	38.499

TABLE I. 2

Eigenvalues for Problem 2, $\alpha = 1/3$

n	$\lambda_n(\text{exact})^1$	$\lambda_n(\text{Eq. 3.15})^3$	$\lambda_n(\text{Eq. 5.3})$	n	$\lambda_n(\text{exact})^1$	$\lambda_n(\text{Eq. 3.15})^3$	$\lambda_n(\text{Eq. 5.3})$
0	1.572	1.833	1.555	5	10.96	11.833	10.958
1	3.385	3.833	3.375	6	12.88	13.833	12.881
2	5.25	5.833	5.244	7	14.81	15.833	14.810
3	7.14	7.833	7.136	8	16.74	17.833	16.743
4	9.05	9.833	9.042	9	18.68	19.833	18.680

TABLE I. 3

Eigenvalues for Problem 3, $a = 1/3$

n	$\lambda_n(\text{exact})^a$	$\lambda_n(\text{Eq. }3.15)^3$	$\lambda_n(\text{Eq. }5.3)$	n	$\lambda_n(\text{Exact})^a$	$\lambda_n(\text{Eq. }3.15)^3$	$\lambda_n(\text{Eq. }5.3)$
0	1	1.167	0.979	10	19.978	21.167	19.973
1	2.774	3.167	2.760	11	21.920	23.167	21.915
2	4.629	5.167	4.618	12	23.864	25.167	23.859
3	6.513	7.167	6.504	13	25.811	27.167	25.806
4	8.414	9.167	8.406	14	27.759	29.167	27.754
5	10.326	11.167	10.319	15	29.709	31.167	29.704
6	12.247	13.167	12.240	16	31.661	33.167	31.656
7	14.173	15.167	14.166	17	33.614	35.167	33.610
8	16.104	17.167	16.098	18	35.569	37.167	35.565
9	18.040	19.167	18.034	19	37.525	39.167	37.520

[a] These exact results have been calculated by the present author. The first 10 agree
with Fox and Libby, Part 2, to within 0.01

TABLE II

Norms for Problem 3, $a = -1/3$

n	$C_n(\text{exact})$	$C_n(\text{Eqs. 3.17})^3$	n	$C_n(\text{exact})$	$C_n(\text{Eqs. 3.17})^3$
0	2.268	2.035	10	6.077	6.00
1	3.184	3.045	11	6.262	6.19
2	3.772	3.650	12	6.438	6.36
3	4.219	4.11	13	6.604	6.53
4	4.586	4.49	14	6.763	6.69
5	4.903	4.81	15	6.915	6.84
6	5.182	5.09	16	7.059	6.99
7	5.435	5.35	17	7.198	7.12
8	5.665	5.58	18	7.333	7.26
9	5.878	5.80	19	7.461	7.39

more accurate, since the method of Nachtsheim and Swigert[8] was used
to obtain them, and the criterion for stopping the integration was the
constancy of C_n itself as $\eta \rightarrow \infty$.)

VI.   THE INSULATED WALL PROBLEM

Problem 3, a flat plate with constant wall condition g_{w1} up to ξ_i and insulated ($g_{\eta w} = 0$) for $\xi > \xi_i$, has had some previous attention. In particular, Durgin[9, 10] made a thorough experimental and theoretical study, reviewing the previous solutions, all of which were approximate. He concluded that the laminar experimental results for wall temperature distribution in the insulated region were in very good agreement with the theory based on Lighthill's approximate method,[4] thus giving it credence as the best approximation to the exact solution. The solution of Fox and Libby,[1] given in (4.2), is an exact solution expressed in series form and may be compared with the Lighthill-Durgin solution.

Strangely enough, no numerical solution to the partial differential equation of the problem seems to have been published up to this time, although such a solution would represent another exact solution to serve as a standard of comparison not only for the Lighthill-Durgin approximation, but for the rate of convergence of the Fox-Libby series.

Such a numerical solution has been obtained as a by-product of a computer program developed for another problem.[11] It solves the energy equation derived from (2.2) when $f = f_o(\eta)$:

$$g_{\eta \eta} + f_o g_\eta - 2 \xi \, f'_o \, g_\xi = 0 \qquad\qquad (6.1)$$

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The boundary and initial conditions are:

$$\xi > \xi_i: \qquad g_\eta \ (\eta = 0) = 0, \ g(\eta \rightarrow \infty) \rightarrow 1 \qquad\qquad (6.\,2a)$$

$$\xi = \xi_i: \qquad g_i = g_{w1} + (1-g_{w1}) \, f_o' \qquad\qquad (6.\,2b)$$

The parameter g_{w1} is eliminated by the transformation

$$G = (g-1) \, / \, (g_{w1} - 1) \qquad\qquad (6.\,3)$$

and the streamwise coordinate is simplified by introducing a new coordinate

$$z = \ln \, (\xi/\xi_i)^{1/2}, \qquad z \geq 0 \text{ for } \xi \geq \xi_i \qquad\qquad (6.\,4)$$

Then (6.1) and (6.2) become

$$G_{\eta\eta} + f_o \, G_\eta - f_o' \, G_z = 0 \qquad\qquad (6.\,5)$$

$$z > 0: \ G_\eta \ (\eta = 0) = 0, \ G \ (\eta \rightarrow \infty) = 0 \qquad\qquad (6.\,6a)$$

$$z = 0: \ G_i = 1 - f_o' \qquad\qquad (6.\,6b)$$

This is a problem with no parameters. The new streamwise coordinate z seems to be the natural one for initial value problems (where $\xi_i \neq 0$) because it takes the operator $2 \, \xi \, \partial \, / \, \partial \ \xi$ into $\partial \, / \, \partial \ z$. It is rather surprising that it does not seem to have had any widespread use in numerical solutions of boundary layer problems.

The solution to (6.5), (6.6) was obtained by use of a Crank-Nicolson implicit finite difference scheme. [12] The η derivatives are averages of central difference formulas at the known station z and the new station $z + \Delta z$, while the z derivative is a forward difference. This

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leads at each z to a set of linear inhomogenous algebraic equations, each of which contains the unknown at only 3 successive η points. The matrix of coefficients is in tri-diagonal form, and the equations are easily solved by successive elimination, starting at $\eta = 0$, moving to the edge of the boundary layer, and then sweeping back toward the wall again. The algorithm is easily formalized.[12]

The boundary condition $G = 0$ at $\eta \rightarrow \infty$ is satisfied at a finite value η_e, chosen so that f_o' is very near its asymptotic value of unity. We used $\eta_e = 4.8$ as the edge. The wall condition is on G_η which presents some complications, since the central difference at the wall involves a point inside, while that one step off the wall involves the unknown G_w. But the differential equation (6.5) at the wall, and the boundary condition (6.6a) together tell us that

$$G_{\eta w} = 0, \quad G_{\eta \eta w} = 0$$

Therefore, a Taylor series expansion from the wall $\eta = 0$ to the first station $\eta = \Delta \eta$ shows that

$$G_w = G(\Delta \eta) + O(\Delta \eta^2)$$

To the order of accuracy of the central difference formulas then, $G_w = G(\Delta \eta)$, and this relation makes the algebraic solution of the difference equations determinate, if we start with the equation centered at $\eta = \Delta \eta$ and end with the one at $\eta = \eta_e - \Delta \eta$, taking $G(\eta_e) = 0$.

By this scheme, solutions have been obtained with $\eta_e = 4.8$, $\Delta \eta = 0.05$, for $\Delta z = 5 \times 10^{-3}$, 5×10^{-4}, and 5×10^{-5}. The coefficient functions, f_o, f_o' were generated first, by solving (2.3) with $f_o''(0) = a = 0.4696$ with the same $\Delta \eta$. The results for G_w are shown in Fig. 2 over the range $0 \leq z \leq 0.13$.

ANALYTICAL
× FOX & LIBBY[1] : 10 TERMS
△ LIBBY & MORDUCHOW[13]
+ LIGHTHILL - DURGIN[9,10]
◆ FOX & LIBBY : 20 TERMS

NUMERICAL:
◊ $\Delta z = 5 \times 10^{-5}$
○ $\Delta z = 5 \times 10^{-4}$
□ $\Delta z = 5 \times 10^{-3}$

G_W

$$G_w = \frac{g_w - 1}{g_{w1} - 1}$$

0.9

0.8

0.7

0.6

0.5

1

0

0   .01   .02   .03   .04   .05   .06   .07   .08   .09   .10   .11   .12
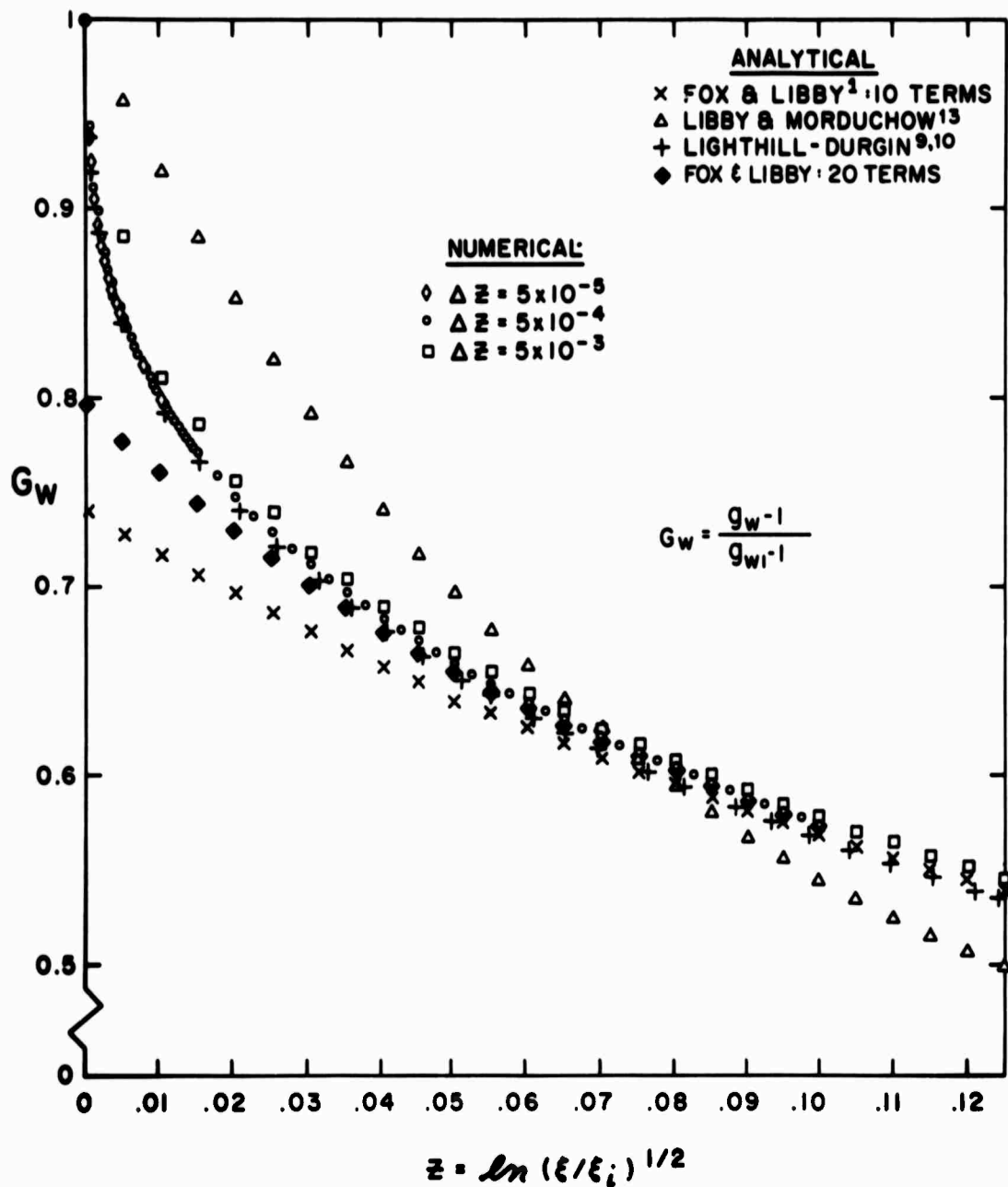
$$z = \ln (\xi/\xi_i)^{1/2}$$

Fig. 2   Wall enthalpy parameter for an insulated wall following a wall of constant enthalpy ratio g_{w1}.

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A9891

which corresponds to $1 \leq \xi/\xi_i \leq 1.3$. This short distance near the initial

plane has been singled out to study, because the behavior near the initial

plane is the most stringent test of any approximate solution.

The numerical solutions seem to approach $G_w = 1$ smoothly, but

with a very steep slope, which can well be interpreted as infinite at $z = 0$.

The Fox-Libby series, (4.2), is also shown through 10 terms by

the X symbols. They show good agreement with the numerical solutions

(within 10%) for $z \geq 0.09$, but fall below it for smaller z, reaching only

$G_w = 0.7426$ at $z = 0$. (This solution was calculated using the exact

eigenvalues given in Table I.3, and the exact norms shown in Table II.)

Extending the series to 20 terms yields the results shown on Fig. 2 by the

solid diamonds. The solution is improved, but still only reaches $G_w = 0.8$

at $z = 0$. Clearly, many terms are needed to obtain accuracy near the

initial plane because of the slow convergence of the series (4.2), although

we proved in Section V that it did converge and could presumably yield

$G_w = 1$ at $z = 0$ eventually.

The solution to this problem based on Lighthill's method was worked

out by Durgin.[9] The velocity profile is approximated by a linear function

$f'_o = a\eta$, VonMises variables are used, and the equation is solved by a

Laplace Transform in the streamwise coordinate. One may then invert

back to the physical plane in two ways. Lighthill[4] finds heat transfer as

an integral over the wall temperature distribution. For the present

problem we want the wall temperature as an integral over the heat

transfer. Into this integral we substitute the shear, and the heat

transfer of the similar boundary layer over the front of the plate $0 < \xi < \xi_i$.

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For this latter heat transfer we use the value obtained from the Lighthill approximation for a constant temperature wall, to be consistent with the approximation being used for $\xi > \xi_i$. The result of integration is then, for $\xi > \xi_i$,

$$G_w = \frac{\sin \pi/3}{\pi} \int_0^{(\xi_i/\xi)^{3/4}} \frac{du}{u^{1/3} (1-u)^{2/3}} \tag{6.7}$$

which was given by Durgin. [9]

While this is an incomplete Beta function, its tabulation can be easily accomplished by using Simpson's rule on the integral, after removing the singularity of the integrand. To accomplish this we let

$$\xi/\xi_i = (\cos \theta)^{-8/3} \text{ and find}$$

$$G_w = 1 - \frac{2 \sin \pi/3}{\pi} \left\{ \frac{3\theta^{2/3}}{2} + \int_0^\theta \left[(\cot\phi)^{1/3} - \phi^{-1/3} \right] d\phi \right\} \tag{6.8}$$

The result of such a calculation is shown in Fig. 2 as the + symbols. They show remarkable agreement with the numerical solution right up to $z = 0$, including the rapid increase in slope. From (6.7) the slope near $z = 0$ is

$$\frac{dG_w}{dz} = - \frac{(3/2)^{1/3} \sin \pi/3}{\pi z^{2/3}} = - \frac{0.316}{z^{2/3}} \tag{6.9}$$

which shows the strength of the singularity.

As $\xi/\xi_i \to \infty$, we find from (6.7) that

$$G_w \to \frac{3 \sin \pi/3}{2\pi} \left(\frac{\xi}{\xi_i} \right)^{-1/2} = 0.414 \left(\frac{\xi}{\xi_i} \right)^{-1/2}$$

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which may be compared with the corresponding result from the Fox-Libby series, which is just the first term, i. e.

$$G_w \rightarrow (\xi/\xi_i)^{-1/2}/2.268 = 0.440 \, (\xi/\xi_i)^{-1/2} \tag{6.10}$$

Since the latter result is presumably exact, we see the Lighthill-Durgin solution is some 6% low for large ξ. So its accuracy is likely to be high over the whole range.

Durgin already found (6.7) to be in good agreement with his experimental results, and now we have shown it to be also in good agreement with our exact numerical solution. This leaves little doubt that the simple integral (6.7) is the most useful available solution to the present problem. One may speculate on why a solution developed for large Prandtl number should be so accurate for unity Prandtl number. We already know that the Lighthill approximation for heat transfer predicts nearly the correct Prandtl number dependence, the $1/3$ power, although a somewhat erroneous constant. But the relation leading to (6.7) involves the heat transfer divided by the $1/3$ power of the Prandtl number, so in fact, (6.7) is obtained as independent of the Prandtl number, and holds for any value of that parameter. It is likely to give a good answer when compared with numerical solutions obtained for nonunity Prandtl numbers. (In fact, Durgin's experiments were for air, which has a Prandtl number about 0.72.) One must only be careful to define G_w to be $(T_w - T_{aw})/(T_{w1} - T_{aw})$ where T_{aw} is the adiabatic wall temperature, so the correct limit is approached for downstream. In the present case of unit Prandtl number, T_{aw} is the same as the free stream stagnation temperature, so our definition of G_w, (6.3), is consistent with this.

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As an example of integral-type solutions, we may quote that of Libby and Morduchow, [13] which Fox and Libby also plot in their Part 2, Fig. 5, and Durgin[9] quotes. For Prandtl number unity and constant density-viscosity product, this result is

$$G_w = 0.476 \ (\xi/\xi_i)^{-1/2} + 0.524 \ (\xi/\xi_i)^{-7.63}$$

This is also plotted as the triangle symbols in Fig. 2. It clearly has the wrong behavior near $z = 0$, and is 8% high as $\xi \to \infty$ according to (6.10). In the intermediate range, starting about $\xi/\xi_i = 2.5$, it is in rather good agreement with 10 terms of (4.2), better than Fox and Libby show in their figure, where they seem to have an error in the Libby-Morduchow curve. But in any event, it is less accurate than (6.7), and hardly any more convenient.

VII. CONCLUSIONS

By pursuing further Kotorynski's uniform approximation[3] it has been possible to obtain explicit expressions for the eigenfunctions and their norms in three boundary layer eigenvalue problems. The convergence of the Fox-Libby eigenfunction series solution to two heat transfer problems has then been investigated, using these norms and the eigenvalues obtained by Kotorynski. The series both converge away from the initial plane with a convergence factor ξ_i/ξ, but at the initial plane, the series for a discontinuous wall temperature diverges. The series for an insulated wall following a constant temperature wall converges at the initial plane, but its streamwise derivative diverges there.

We have shown that the approximate eigenvalues obtained by Kotorynski can be greatly improved by using the WKB method. This improvement is obtained by using a more accurate approximation to the Blasius function for large η, i.e., using $f_o \rightarrow \eta-\beta$ instead of $f_o \rightarrow \eta$. While the Kotorynski type of explicit solution cannot be obtained if β is kept, the WKB method for finding the eigenvalues can be easily applied. The resulting eigenvalues are found as the solution of a quadratic equation, and are very accurate. Their use enables new eigenfunctions to be found by a simple forward integration of the differential equation, with little, if any, iteration necessary.

Finally, we have studied the insulated wall problem using the eigenfunction series, a quadrature formula developed by Durgin[9] based on Lighthill's approximation,[4] an integral method of Libby and Morduchow,[13] and an exact

numerical solution of the partial differential equation obtained by the present author. [11] Comparison is made near the initial plane, where the numerical solution has a very steep slope as it approaches the constant wall tempera- ture value. The Lighthill-Durgin solution agrees almost perfectly with the numerical solution in this region, having a slope proportional to $(\xi/\xi_i - 1)^{-2/3}$. The Fox-Libby series differs considerably near the initial plane, even with 20 terms, attesting to its slow convergence there. The integral solution also is inaccurate there.

As $\xi \rightarrow \infty$ far downstream, the Lighthill-Durgin solution and the inte- gral solution both have the same decay as the (presumably exact) series solution, but the former is 6% lower and the latter 8% higher than the series.

We conclude that the Lighthill-Durgin quadrature formula is an excel- lent approximation to the exact solution over the whole range. This is true not only at Prandtl number unity, but also at other values, since this solu- tion is independent of that parameter when referred to the adiabatic wall temperature. This result is in accord with Durgin's experimental work, [9,10] which was performed in air, with Prandtl number 0.72. He found the Lighthill- Durgin solution to be in very good agreement with his laminar data.

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DOCUMENT CONTROL DATA - R&D
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Avco Everett Research Laboratory 2385 Revere Beach Parkway Everett, Massachusetts	Unclassified
	2b GROUP

3. REPORT TITLE

Approximate Solution of Boundary Layer Heat Transfer
Eigenvalue Problems with Applications

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Research Report 337

5. AUTHOR(S) (Last name, first name, initial)

Kemp, Nelson H.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1970	40	13

8a. CONTRACT OR GRANT NO. F04701-68-C-0036 b. PROJECT NO. ARPA Order #1092	9a. ORIGINATOR'S REPORT NUMBER(S) Research Report 337
c	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) SAMSO-TR-69-347
d	

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13 ABSTRACT

We study three eigenvalue problems of boundary layer theory first
introduced by Fox and Libby,[1] and pursued further by Kotorynski.[2] By
completing Kotorynski's approximate analysis, we exhibit approximate
eigenvalues and norms. With these, the convergence of the eigenfunction
series solutions of two heat transfer problems can be investigated: (a) Flat
plate with one constant wall temperature region followed by another.
(b) Flat plate with a constant wall temperature region followed by an in-
sulated wall. The series for problem (b) is found to converge everywhere,
including at the station of change in the boundary condition. The series
for problem (a) does not converge at that station, and convergence for both
appears slow near that station. Then we improve Kotorynski's approxima-
tion and use the WKB method to obtain simple formulas for the eigenvalues,
which prove very accurate, and enable the eigenfunctions to be found by a
simple forward integration. Finally, we study problem (b) in more detail,
and obtain an exact solution by a numerical finite difference procedure.
This shows the eigenfunction series solution to be very slowly convergent
near the beginning of the insulated region. The solution worked out by
Durgin[9,10] based on Lighthill's[4] approximate method, is very accurate
there. In addition, it is within 10% of the numerical solution everywhere.
It is undoubtedly the most convenient representation of the solution, agree-
ing both with the new numerical solution, and with Durgin's experiments.

DD FORM 1473
1 JAN 64

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
1. Boundary Layer 2. Heat Transfer 3. Analytical Solution 4. Numerical Solution 5. Eigenvalues 6. Eigenfunctions 7. WKB Method						

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